observations extended, it is easy to calculate approximately the correction required by Leverrier's value of the parallax. Assuming the average tabular parallax of *Mars* during the Ascension observations to have been $21''$, and taking Mr. Gill's constant $''\cdot376$ as the correction to be deducted from this value, we have for the correction to the Sun's tabular parallax of $8''\cdot95$,

$$-\frac{''\cdot376 \times 8\cdot95}{21\cdot00} = -''\cdot16.$$

Thus, we get $8''\cdot95 - ''\cdot16 = 8''\cdot79$, agreeing extremely well with the general result of Mr. Gill's observations.

I have for some time thought that an extension of the method employed by Leverrier in the case of the Sun to determine the parallax of some of the nearer planets, would probably be attended with success, and the result brought out incidentally by Mr. Gill seems to support this view. One of the main difficulties would be the accurate determination of the Moon's mass. Probably, sufficiently numerous and exact observations exist to determine the lunar parallax to within $\frac{1}{100000}$th of its actual value, *i.e.* to about $''\cdot03$, while the length of the seconds pendulum may be considered as already known with an equal exactitude; these two data should therefore give us the value of the Moon's mass to within the $\frac{1}{1000}$th part of the truth. As respects other methods of determining the solar parallax, it appears to me that something more might probably be accomplished by a careful analysis of existing observations of *Jupiter's* satellites, treating the diameters of the planet and of the satellites as unknown quantities to be determined from the eclipse observations, and taking into account the possible effect of the refraction of the Jovian atmosphere (which may be considerable) on the times of disappearance and reappearance.

Stroud Green, N.,
1881, *July* 13.

On the Determination of the Moon's Libration.
By A. Marth, Esq.

In the method which has been adopted in several investigations of the Moon's libration, the apparent position of a spot referred to the centre of the Moon's disk and to the circle of declination is deduced from the measured distances and position-angles of a number of points of the Moon's limb in reference to the spot; from the apparent coordinates thus found the selenocentric right ascension and declination of the spot are deduced, and these are converted into ecliptical longitude and latitude. Comparison is then made with the selenocentric longitude and

latitude computed from approximate values of the spot's selenographical coordinates and of the inclination of the lunar equator to the ecliptic, and the equations are found which exhibit the effect of corrections of these values and of the terms due to physical libration.

It appears preferable to follow another course; to proceed from assumed selenographical coordinates of the spot to its apparent coordinates referred to the centre of the Moon's disk and the direction of the Moon's axis, to determine the corrections of these coordinates which best satisfy the measurements, and to form the equations between the corrections and the effect of the Moon's physical libration, &c., upon these coordinates.

Let I denote the assumed inclination of the plane of the lunar equator to the ecliptic; ☊ $+ 180°$ the longitude of its ascending node, which coincides with the descending node of the orbit; J its inclination to the plane of the Earth's equator; N the right ascension of the ascending node. If, further, l_0 is the mean longitude of the Moon in its orbit and $M + 180°$ the arc of the lunar equator between equator and ecliptic, let $l_0 -$ ☊ $+ M + N$ be denoted by α_0. The departure of the first or zero meridian, from which the selenographical longitudes are reckoned, from the ascending node ☊ $+ 180°$ of the lunar equator on the ecliptic, is, in consequence of the equality of the periods of rotation and mean revolution, $= l_0 -$ ☊, and therefore its departure from the ascending node of the lunar equator on the Earth's equator

$$= l_0 - \text{☊} + M + 180° = \alpha_0 - N + 180°,$$

so that the angle α_0 is in reference to the equator analogous to l_0 in reference to the ecliptic.

Ephemerides of the position of the lunar equator referred to that of the Earth have been given in the *Berliner Jahrbuch* since 1839, and in the *Nautical Almanac* since 1867, the notation and formulæ adopted being those employed by Encke in his paper "Ueber die selenocentrischen Constanten bei den Sternbedeckungen und die Berechnung der Libration des Mondes, nebst Tafeln," in the *Berliner Jahrbuch* for 1843. Instead of the arc of the lunar equator between the Earth's equator and the ecliptic, therein given, it will be found far more convenient to have the value of α_0, which is in Encke's notation

$$= l_0 - \text{☊} + 180° + \Delta + \text{☊}'.$$

For the present purpose the Ephemerides will have to be computed rather more accurately, l_0 and ☊ being duly referred to the true equinox. Since, moreover, other formulæ will be found preferable to Encke's, there would be no advantage in retaining his designations i, Δ and ☊$'$, which are replaced by the letters J, $M + 180°$ and N in the present paper.

The values of α_0, N and J may be deduced from those of l_0,

☊, I, and of the apparent obliquity ϵ of the ecliptic by means of the formulæ

$$\tan\eta = -\tan I \cos ☊,$$
$$\tan N = -\tan I \sin ☊ \cdot \frac{\cos\eta}{\sin(\epsilon+\eta)},$$
$$\tan J = \tan(\epsilon+\eta)\sec N,$$
$$\sin\tfrac{1}{2}(\alpha_\circ - l_\circ) = -\frac{\sin\frac{1}{2} I \sin\frac{1}{2}\epsilon}{\cos\frac{1}{2} J} \cdot \sin ☊.$$

The next proceeding is the computation of the position-angle P of the Moon's axis, and of the selenographical longitude Λ and latitude B of the centre of the Moon's disk as seen by the observer. Let α' and δ' be the apparent right ascension and declination of the Moon's centre, which are derived from the geocentric right ascension and declination α and δ, if θ is the sidereal time, by

$$r'\cos\delta'\sin(\alpha'-\alpha) = -\rho'\sin(\theta-\alpha),$$
$$r'\cos\delta'\cos(\alpha'-\alpha) = r\cos\delta - \rho'\cos(\theta-\alpha),$$
$$r'\sin\delta' = r\sin\delta - \rho'',$$

or by equivalent formulæ, in which $r(=\frac{1}{\sin\pi})$ and r' are the linear distances of the Moon's centre from the Earth's centre and from the observer, and ρ' and ρ'' the perpendicular distances of the observer from the Earth's axis and from the plane of the equator, the unit of these distances being the equatoreal semi-diameter of the Earth. The selenocentric spherical triangle between the pole of the lunar equator, the pole of the celestial equator and the observer furnishes the equations:

$$\cos B\sin P = -\sin J\cos(\alpha'-N),$$
$$\cos B\cos P = +\sin J\sin(\alpha'-N)\sin\delta' + \cos J\cos\delta',$$
$$\sin B = +\sin J\sin(\alpha'-N)\cos\delta' - \cos J\sin\delta',$$
$$\cos B\sin(\Lambda+\alpha_\circ-N) = \cos J\sin(\alpha'-N)\cos\delta' + \sin J\sin\delta',$$
$$\cos B\cos(\Lambda+\alpha_\circ-N) = \cos(\alpha'-N)\cos\delta';$$

from which latter two may be deduced

$$\cos B\sin(\Lambda+\alpha_\circ-\alpha') = [\sin\delta' - \tan\tfrac{1}{2} J\sin(\alpha'-N)\cos\delta']\sin J\cos(\alpha'-N)$$
$$= (\sin\delta' - \sin B)\tan\tfrac{1}{2} J\cos(\alpha'-N).$$

Hence are derived the formulæ for the computation of P, B and Λ:—

$$\tan\zeta = \tan J\sin(\alpha'-N),$$
$$\tan P = -\tan J\cos(\alpha'-N) \cdot \frac{\cos\zeta}{\cos(\zeta-\delta')},$$
$$\tan B = \tan(\zeta-\delta') \cdot \cos P,$$
$$\sin(\Lambda+\alpha_\circ-\alpha') = \frac{\sin B - \sin\delta'}{2\cos^2\frac{1}{2} J} \cdot \sin P,$$
$$\text{or} = \frac{\sin\frac{1}{2}(B-\delta')\cos\frac{1}{2}(B+\delta')}{\cos^2\frac{1}{2} J} \cdot \sin P,$$
$$\text{or} = \frac{\sin(\zeta'-\delta')}{\cos\zeta'} \cdot \sin P \quad, \text{ if } \tan\zeta' = \tan\tfrac{1}{2} J\sin(\alpha'-N).$$

Then follows the computation of the apparent position of the spot. If λ and β are the selenographical longitude and latitude of the spot, and p its position-angle at the centre of the disk, as seen by the observer, we have selenocentrically, if the angular distance between spot and observer is called S,

$$\begin{aligned}
\sin S \sin (p-P) &= \cos \beta \sin (\Lambda-\lambda),\\
\sin S \cos (p-P) &= -\cos \beta \cos (\Lambda-\lambda) \sin B + \sin \beta \cos B,\\
\cos S &= \cos \beta \cos (\Lambda-\lambda) \cos B + \sin \beta \sin B.
\end{aligned}$$

If the linear distance of the spot from the Moon's centre of gravity is k' and from the observer r'', the unit of distances being, as before, the semi-diameter of the Earth's equator, the angular distance s of the spot from the centre of the disk, as seen by the observer, and r'' are found by

$$r'' \sin s = k' \sin S,$$
$$r'' \cos s = r' - k' \cos S;$$

or, indirectly, by

$$\sin s = \frac{k'}{r'} \sin (S+s),$$
$$r'' = r' \cdot \frac{\sin S}{\sin (S+s)}.$$

The coordinates

$$x'' = \omega'' \sin s \sin (p-P), \qquad \left(\omega'' = \frac{1}{\text{arc } 1''} = 206264''\cdot 8\right),$$
$$y'' = \omega'' \sin s \cos (p-P),$$

or the apparent rectangular coordinates of the spot in reference to the centre of the disk and to the computed direction of the Moon's axis seem to me the most suitable for the determination of the libration. The corrections of x'' and y'' which best satisfy the observations, compared with the variations of x'' and y'' which represent the effect of the corrections of the assumed values of I, λ, β, and of the different terms of the physical libration, furnish then the equations of condition for the determination of the latter.

In the differentiation of the equation for x'' and y''

$$x'' = \frac{\omega'' k'}{r''} \cdot \sin S \sin (p-P) = \frac{\omega'' k'}{r''} \cdot \cos \beta \sin (\Lambda-\lambda),$$

$$y'' = \frac{\omega'' k'}{r''} \cdot \sin S \cos (p-P) = \frac{\omega'' k'}{r''} [-\cos \beta \cos (\Lambda-\lambda) \sin B + \sin \beta \cos B],$$

it must be borne in mind that P as a function of the assumed

elements is itself variable, so that if the corrected values $x''+\delta x''$ and $y''+\delta y''$ are to refer to the same direction P as x'' and y'' themselves, the variation of P must be duly taken into account. Differentiating accordingly, we get

$$\delta x'' = \frac{k'}{r''}\left\{ \begin{array}{l} \cos\beta\cos(\Lambda-\lambda)\,\delta(\Lambda-\lambda) \\ -\sin\beta\sin(\Lambda-\lambda)\,\delta\beta \\ +\sin \mathrm{S}\cos(p-\mathrm{P})\,\delta\mathrm{P} \end{array}\right\} + x''\,.\,\frac{\delta k'}{k'},$$

$$\delta y'' = \frac{k'}{r''}\left\{ \begin{array}{l} \cos\beta\sin\mathrm{B}\sin(\Lambda-\lambda)\,\delta(\Lambda-\lambda) \\ +[\cos\beta\cos\mathrm{B}+\sin\beta\sin\mathrm{B}\cos(\Lambda-\lambda)]\,\delta\beta \\ -[\cos\beta\cos\mathrm{B}\cos(\Lambda-\lambda)+\sin\beta\sin\mathrm{B}]\,\delta\mathrm{B} \\ -\sin\mathrm{S}\sin(p-\mathrm{P})\,\delta\mathrm{P} \end{array}\right\} + y''\,.\,\frac{\delta k'}{k'}.$$

From the expressions of Λ, B, P are derived

$$\begin{aligned} \delta\Lambda &= -\delta\alpha_\circ - [\tan\mathrm{B}\sin(\alpha_\circ-\mathrm{N}+\Lambda)-\tan\tfrac{1}{2}\mathrm{J}]\sin\mathrm{J}.\delta\mathrm{N} \\ &\quad -\tan\mathrm{B}\cos(\alpha_\circ-\mathrm{N}+\Lambda)\,\delta\mathrm{J}, \\ \delta\mathrm{B} &= -\cos(\alpha_\circ-\mathrm{N}+\Lambda)\sin\mathrm{J}.\delta\mathrm{N}+\sin(\alpha_\circ-\mathrm{N}+\Lambda)\,\delta\mathrm{J}, \\ \cos\mathrm{B}.\delta\mathrm{P} &= -\sin(\alpha_\circ-\mathrm{N}+\Lambda)\sin\mathrm{J}.\delta\mathrm{N}-\cos(\alpha_\circ-\mathrm{N}+\Lambda)\,\delta\mathrm{J}. \end{aligned}$$

But, if

$$\mathrm{M} = \alpha_\circ - \mathrm{N} - (l_\circ - \Omega),$$

then

$$\begin{aligned} \delta\alpha_\circ &= \delta l_\circ - (\tan\tfrac{1}{2}\mathrm{J}\cos\mathrm{M}+\tan\tfrac{1}{2}\mathrm{I})\sin\mathrm{I}.\delta\Omega \\ &\quad -\tan\tfrac{1}{2}\mathrm{J}\sin\mathrm{M}.\delta\mathrm{I}, \\ \sin\mathrm{J}.\delta\mathrm{N} &= -\cos\mathrm{M}\sin\mathrm{I}.\delta\Omega-\sin\mathrm{M}.\delta\mathrm{I}, \\ \delta\mathrm{J} &= +\sin\mathrm{M}\sin\mathrm{I}.\delta\Omega-\cos\mathrm{M}.\delta\mathrm{I}. \end{aligned}$$

Hence

$$\begin{aligned} \delta\Lambda &= -\delta l_\circ + [\tan\mathrm{B}\sin(l_\circ-\Omega+\Lambda)+\tan\tfrac{1}{2}\mathrm{I}]\sin\mathrm{I}.\delta\Omega \\ &\quad +\tan\mathrm{B}\cos(l_\circ-\Omega+\Lambda)\,\delta\mathrm{I}, \\ \delta\mathrm{B} &= \cos(l_\circ-\Omega+\Lambda)\sin\mathrm{I}.\delta\Omega-\sin(l_\circ-\Omega+\Lambda)\,\delta\mathrm{I}, \\ \cos\mathrm{B}.\delta\mathrm{P} &= \sin(l_\circ-\Omega+\Lambda)\sin\mathrm{I}.\delta\Omega+\cos(l_\circ-\Omega+\Lambda)\,\delta\mathrm{I}, \end{aligned}$$

the substitution of which gives

$$\delta x'' = \frac{k' \cos \beta}{r''} \Big\{ - \cos (\Lambda - \lambda) (\delta l_0 + \delta\lambda)$$
$$- \tan \beta \sin (\Lambda - \lambda) \delta\beta$$
$$+ [\tan \beta \sin (l_0 - \Omega + \Lambda) + \tan \tfrac{1}{2} \mathrm{I} \cos (\Lambda - \lambda)] \sin \mathrm{I}.\delta\Omega$$
$$+ \tan \beta \cos (l_0 - \Omega + \Lambda) \delta \mathrm{I} \Big\}$$
$$+ x'' . \frac{\delta k'}{k'},$$

$$\delta y'' = \frac{k' \cos \beta}{r''} \cos \mathrm{B} \Big\{ - \tan \mathrm{B} \sin (\Lambda - \lambda) (\delta l_0 + \delta\lambda)$$
$$+ [1 + \tan \beta \tan \mathrm{B} \cos (\Lambda - \lambda)] \delta\beta$$
$$- [\cos (l_0 - \Omega + \lambda) + \tan \beta \tan \mathrm{B} \cos (l_0 - \Omega + \Lambda)$$
$$- \tan \tfrac{1}{2} \mathrm{I} \tan \mathrm{B} \sin (\Lambda - \lambda)] \sin \mathrm{I}.\delta\Omega$$
$$+ [\sin (l_0 - \Omega + \lambda) + \tan \beta \tan \mathrm{B} \sin (l_0 - \Omega + \Lambda)] \delta \mathrm{I} \Big\}$$
$$+ y'' . \frac{\delta k'}{k'};$$

or, putting

$$\tan \beta \tan \mathrm{B} \sin (\Lambda - \lambda) = \nu \sin (\lambda_1 - \lambda)$$
$$1 + \tan \beta \tan \mathrm{B} \cos (\Lambda - \lambda) = \nu \cos (\lambda_1 - \lambda),$$

then

$$\delta y'' = \frac{k' \cos \beta}{r''} \cos \mathrm{B} \Big\{ - \tan \mathrm{B} \sin (\Lambda - \lambda) (\delta l_0 + \delta\lambda)$$
$$+ \nu \cos (\lambda_1 - \lambda) \delta\beta$$
$$- [\nu \cos (l_0 - \Omega + \lambda_1) - \tan \tfrac{1}{2} \mathrm{I} \tan \mathrm{B} \sin (\Lambda - \lambda)] \sin \mathrm{I}.\delta\Omega$$
$$+ \nu \sin (l_0 - \Omega + \lambda_1) \delta \mathrm{I} \Big\}$$
$$+ y'' . \frac{\delta k'}{k'}.$$

The physical libration produces oscillations in the values of l_0, Ω and I, the theoretical investigation of which is to be found in the *Méc. Cél.* L. v., and in the papers of Poisson (*Conn. des Temps*, for 1821 and 1822), Bessel (*Astr. Nachr.* Nos. 376 and 377, or *Abhandlungen*, iii. pp. 317–328), Wichmann (*Astr. Nachr.* Nos. 619, 621, 628, 630, 631), and Hartwig (*Beitrag zur Bestimmung der physischen Libration des Mondes*, 1880). If i denotes the inclination of the Moon's orbit to the ecliptic, e its eccentricity, ω the angle in the orbit from the node to the perigee, g the Moon's mean anomaly and Σ (H sin h) the sum of the several inequalities of the motion in longitude, which produce sensible terms; if, further, m', n', π' denote the daily motions of the mean longitude, of the node and of the perigee, and h' the daily motion of h, all of which quantities are known: the oscillations of l_0, Ω

and I, due to the physical libration, may be represented by the following equations:—

$$\delta l_{\circ} = \sum\left(-\mathrm{H}\,.\,\frac{\theta\,(1-f)}{1+f\theta}\,.\,\sin h\right) + a_{\circ}\sin(\mathrm{A}_{\circ}+\nu_{\circ}t),$$

$$\begin{aligned}\sin \mathrm{I}\delta\Omega = {}& \sin \mathrm{I}\,\delta l_{\circ} \\ & -\kappa q\cos\omega\sin(l_{\circ}-\Omega) \\ & +\kappa q' f\sin\omega\cos(l_{\circ}-\Omega) \\ & +b_{\circ}\sin[\mathrm{B}_{\circ}+\mu t-(l_{\circ}-\Omega)] \\ & +c_{\circ}\sin(\mathrm{C}_{\circ}+\mu' t)\cos(l_{\circ}-\Omega) \\ & +\frac{2c_{\circ}}{\sqrt{f}}\cos(\mathrm{C}_{\circ}+\mu' t)\sin(l_{\circ}-\Omega),\end{aligned}$$

$$\begin{aligned}\delta\mathrm{I} = {}& -\kappa q\cos\omega\cos(l_{\circ}-\Omega) \\ & -\kappa q' f\sin\omega\sin(l_{\circ}-\Omega) \\ & -b_{\circ}\cos[\mathrm{B}_{\circ}+\mu t-(l_{\circ}-\Omega)] \\ & -c_{\circ}\sin(\mathrm{C}_{\circ}+\mu' t)\sin(l_{\circ}-\Omega) \\ & +\frac{2c_{\circ}}{\sqrt{f}}\cos(\mathrm{C}_{\circ}+\mu' t)\cos(l_{\circ}-\Omega),\end{aligned}$$

in which f, $a_{\circ}$, $\mathrm{A}_{\circ}$, $b_{\circ}$, $\mathrm{B}_{\circ}$, $c_{\circ}$, $\mathrm{C}_{\circ}$ are the seven constants of the problem which are to be determined, and in which θ, κq, $\kappa q'$, $\nu_{\circ}$, μ, μ' have the following values:—

$$\left.\begin{aligned}\theta' &= \frac{-2n'm'}{h'h'}\,.\,\frac{1}{i+1} \\ \theta &= \frac{\theta'}{1-\theta'},\end{aligned}\right\}\ \text{for each of the effective inequalities,}$$

$$\kappa q = \frac{-n'e\mathrm{I}}{\pi'-n'}\,.\left(1+\frac{0{\cdot}0391\ i}{i+1}\right),$$

$$\kappa q' = \frac{-2n'e\mathrm{I}}{\pi'-n'},$$

$$\nu_{\circ} = \sqrt{-2n'm'}\,.\,\frac{1}{i+1}\,.\,\sqrt{1-f},$$

$$\mu = m' - \frac{n'\mathrm{I}}{i+1},$$

$$\mu' = -\frac{4}{3}\,.\,\frac{n'\mathrm{I}}{i+1}\,.\,\sqrt{f}.$$

In order to compute the coefficients of the unknown quantities correctly, the values of I and f employed must be sufficiently near the true ones, or the computation will have to be repeated with corrected values. Instead of f itself, it will be preferable to make the correction δf of the best available value of f, the unknown quantity, or rather δf multiplied by a properly chosen factor κ'', so that in the equation for $\delta l_{\circ}$ the term

$$\sum\left(\frac{\mathrm{H}}{\kappa''}\,.\,\frac{\theta\,(1+\theta)}{(1+\theta f)^2}\sin h\right)\kappa''\delta f$$

is to be added to the term

$$\Sigma\left(-\mathrm{H}\,.\,\frac{\theta\,(1-f)}{1+\theta f}\ \sin h\right),$$

which is to be considered as known.

The substitution of the values of δl_0, $\delta \Omega$, and $\delta \mathrm{I}$, which represent the effect of the physical libration, in the previously given equations for $\delta x''$ and $\delta y''$, furnishes then the equations:

$$\delta x'' = -\frac{\kappa k' \cos\beta}{r''} \times$$

$$\Big\{\Big(\cos(\Lambda-\lambda) - \tan\beta \tan \mathrm{I} \sin(l_0 - \Omega + \Lambda)\Big)\ \Sigma\left(-\frac{\mathrm{H}\cos \mathrm{I}}{\kappa}\,.\,\frac{\theta\,(1-f)}{1+\theta f}\ \sin h\right)$$

$$+\ q \tan\tfrac{1}{2}\mathrm{I} \sin g \cos(\Lambda-\lambda)$$

$$+\ q \tan\beta \cos(\omega+\Lambda)$$

$$+\,(q-q'f)\tan\beta \sin\omega \sin\Lambda\Big\}$$

$$-\frac{\kappa k' \cos\beta}{r''}\Big\{\ +\cos(\Lambda-\lambda)\,.\,\frac{\delta\lambda}{\kappa}$$

$$+\tan\beta\sin(\Lambda-\lambda)\,.\,\frac{\delta\beta}{\kappa}$$

$$-\tan\beta\cos(l_0 - \Omega + \Lambda)\,.\,\frac{\delta \mathrm{I}}{\kappa}$$

$$+\Big[\ [\cos(\Lambda-\lambda) - \tan\beta\tan \mathrm{I}\sin(l_0 - \Omega + \Lambda)]\ \Sigma\left(\frac{\mathrm{H}\cos \mathrm{I}}{\kappa\kappa''}\,.\,\frac{\theta\,(1+\theta)}{(1+\theta f)^2}\sin h\right)$$

$$-\frac{q'}{\kappa''}\tan\beta\sin\omega\sin\Lambda\Big]\kappa''\delta f$$

$$+\Big(\cos(\Lambda-\lambda) - \tan\beta\tan \mathrm{I}\sin(l_0 - \Omega + \Lambda)\Big)\,.\,\frac{a_0}{\kappa}\sin(\mathrm{A}_0 + \nu_0 t)$$

$$+\ \tan\beta\,.\,\frac{b_0}{\kappa}\cos(\mathrm{B}_0 + \mu t + \Lambda)$$

$$-\ \tan\beta\,.\,\frac{2}{\sqrt{f}}\,.\,\frac{\cos\Lambda}{\cos\Lambda_1}\,.\,\frac{c_0}{\kappa}\cos(\mathrm{C}_0 + \mu' t - \Lambda_1)\Big\},$$

$$\text{if } \tan\Lambda_1 = \frac{\sqrt{f}}{2}\tan\Lambda$$

$$+\frac{x''}{100}\,.\,\frac{100\,\delta k'}{k'};$$

and, putting as before

$$\nu \sin(\lambda_1 - \lambda) = +\tan\beta\tan \mathrm{B}\sin(\Lambda-\lambda),$$
$$\nu \cos(\lambda_1 - \lambda) = 1 + \tan\beta\tan \mathrm{B}\cos(\Lambda-\lambda),$$

then

$$\delta y'' = -\frac{\kappa k' \cos\beta}{r''} \cdot \cos B \cdot \nu \times$$

$$\Big\{\left(\tan I \cos(l_0 - \Omega + \Lambda) + \frac{\tan B}{\nu}\sin(\Lambda - \lambda)\right) \sum\left(-\frac{H\cos I}{\kappa} \cdot \frac{\theta(1-f)}{1+\theta f}\sin h\right)$$

$$+ q \tan\tfrac{1}{2} I \sin g \cdot \frac{\tan B}{\nu}\sin(\Lambda - \lambda)$$

$$+ q \sin(\omega + \lambda_1)$$

$$+ (q - q'f)\sin\omega\cos\lambda_1\Big\}$$

$$-\frac{\kappa k' \cos\beta}{r''} \cdot \cos B \cdot \nu \Big\{ + \frac{\tan B}{\nu}\sin(\Lambda - \lambda)\frac{\delta\lambda}{\kappa}$$

$$+ \cos(\lambda_1 - \lambda)\frac{\delta B}{\kappa}$$

$$- \sin(l_0 - \Omega + \lambda_1) \cdot \frac{\delta I}{\kappa}$$

$$+\Big[\left(\tan I \cos(l_0 - \Omega + \Lambda) + \frac{\tan B}{\nu}\sin(\Lambda - \lambda)\right) \sum\left(\frac{H\cos I}{\kappa\kappa''} \cdot \frac{\theta(1+\theta)}{(1+\theta f)^2}\sin h\right)$$

$$+ \frac{q'}{\kappa''}\sin\omega\cos\lambda_1\Big]\kappa''\delta f$$

$$+\left(\tan I \cos(l_0 - \Omega + \Lambda) + \frac{\tan B}{\nu}\sin(\Lambda - \lambda)\right) \cdot \frac{a_0}{\kappa}\sin(A_0 + \nu_0 t)$$

$$+ \frac{b_0}{\kappa}\sin(B_0 + \mu t + \lambda_1)$$

$$+ \frac{\cos\lambda_1}{\cos\lambda_{\prime\prime}} \cdot \frac{c_0}{\kappa}\sin(C_0 + \mu' t - \lambda_{\prime\prime})\Big\},$$

$$\text{if } \tan\lambda_{\prime\prime} = \frac{2}{\sqrt{f}}\tan\lambda,$$

$$+\frac{y''}{100} \cdot \frac{100\,\delta k'}{k'}.$$

The factor κ, which is introduced on account of obvious convenience, is arbitrary, and may be put $=240$ or, perhaps better, $=300$. The most convenient value of κ'', the factor of δf, will be that of the largest of the coefficients

$$\frac{H}{\kappa} \cdot \frac{\theta(1+\theta)}{(1+f\theta)^2},$$

or that dependent on the Moon's annual equation.

I have written out the equations for $\delta x''$ and $\delta y''$ in full, retaining all the terms in symbolical form, so as to leave the question of the best numerical values which ought to be employed an open one. The substitution of numerical terms and the introduction of suitable abbreviations will, of course, greatly simplify the equations, especially in case the selected spot is near the lunar equator, a case the advantages of which are apparent at a glance.

The measures from which the observed values of $\delta x''$ and

y'' are to be deduced furnish a series of angular distances $s_1 \ldots s_n$ of the spot from points of the Moon's limb in various position-angles $p_1 \ldots p_n$, the ccrresponding times being $t_1 \ldots t_n$. For these times the values of

$$\alpha', \delta', r', \mathrm{R}', \mathrm{P}, \mathrm{B}, \Lambda, x'', y''$$

(R′ being the Moon's apparent semi-diameter, $\sin \mathrm{R}' = \frac{k}{r'}$), may be found by interpolation from the values of these quantities computed at intervals of, say, 20 or 30 minutes for the extent of each set of measurements. From the distances and position-angles, duly cleared from instrumental errors and from the effects of refraction, the corresponding semi-diameters $\mathrm{R}_1 \ldots \mathrm{R}_n$ are to be derived by means of the equations

$$s_1 \sin (p_1 - \mathrm{P}_1) + x_1'' = \mathrm{R} \sin \mathrm{Q}_1,$$
$$s_1 \cos (p_1 - \mathrm{P}_1) + y_1'' = \mathrm{R}_1 \cos \mathrm{Q}_1,$$
$$\cdots\cdots$$
$$s_n \sin (p_n - \mathrm{P}_n) + x_n'' = \mathrm{R}_n \sin \mathrm{Q} ,$$
$$s_n \cos (p_n - \mathrm{P}_n) + y_n'' = \mathrm{R}_n \cos \mathrm{Q}_n,$$

in which $\mathrm{Q}_1 \ldots \mathrm{Q}_n$ are the position-angles of the observed points of the limb at the centre of the disk, reckoned from the direction of the Moon's axis.

The comparison of these $\mathrm{R}_1 \ldots \mathrm{R}_n$ derived from observation, with the interpolated values $\mathrm{R}'_1 \ldots \mathrm{R}'_n$ supplies the equations of condition

$$\mathrm{R}_1' - \mathrm{R}_1 = \sin \mathrm{Q}_1 \, \delta x'' + \cos \mathrm{Q}_1 \, \delta y'' - \frac{\mathrm{R}'}{\mathrm{R}_0} \delta \mathrm{R}_0,$$
$$\cdots\cdots$$
$$\mathrm{R}'_n - \mathrm{R}_n = \sin \mathrm{Q}_n \, \delta x'' + \cos \mathrm{Q}_n \, \delta y'' - \frac{\mathrm{R}'}{\mathrm{R}_0} \delta \mathrm{R}_0,$$

from which the corrections $\delta x''$, $\delta y''$ and $\delta \mathrm{R}_0$ for each set of measures are to be deduced. The correction $\delta \mathrm{R}_0$ is that of some conveniently chosen mean value R_0, and, in case the tabular parallaxes can be depended on, is common to all the sets of measures made with the same instrument under similar circumstances.

The substitution of the corrections $\delta x''$ and $\delta y''$ derived from observation in the previously given equations leads to the equations of condition for the determination of $\delta\lambda$, $\delta\beta$, $\delta \mathrm{I}$, δf, and the others terms of the physical libration.

In order to be able to trace the influence of the irregularities of the Moon's limb, it will be well to compute the selenographical positions of the several points of the limb, which have been measured in position-angle and distance in reference

to the spot. Their selenographical longitudes λ_n and latitudes β_n are derived from the corresponding values of Q_n, Λ_n, B_n by means of the equations

$$\begin{aligned}
\cos\beta_n \sin(\lambda_n - \Lambda_n) &= -\sin Q_n, \\
\cos\beta_n \cos(\lambda_n - \Lambda_n) &= -\cos Q_n \sin B_n + \tan R_n \cos B_n, \\
\sin\beta_n \qquad &= +\cos Q_n \cos B_n + \tan R_n \sin B_n,
\end{aligned}$$

which in practice, where great accuracy is not required, may be sufficiently simplified.

Although I have left the question of the best numerical values which ought to be substituted in the formulæ an open one, some remarks which bear upon it and touch a few of the practical difficulties of the investigation will not be out of place.

An uncertainty in the determination of I is occasioned by the difficulty of disentangling δI and $b_\circ$, in case the observations do not extend over a sufficiently long period. The angle $B_\circ + \mu t$ is

$$\begin{aligned}
&= B_\circ + m't - \frac{n'I}{i+I}t, \\
&= B_\circ - (l_\circ - \Omega)_\circ + l_\circ - \Omega + \frac{n'i}{i+I} \cdot t,
\end{aligned}$$

if $(l_\circ - \Omega)_\circ$ is the value of $l_\circ - \Omega$ for the time $t=0$, from which t is reckoned; or, putting

$$B_\circ - (l_\circ - \Omega)_\circ = B_{,}; \text{ and}$$

$$\frac{n'i}{i+I} = \mu_{,},$$

$$B_{,} + \mu t = B_{,} + l_\circ - \Omega + \mu_{,}t.$$

Hence in the equation for $\delta y''$

$$\begin{aligned}
b_\circ \sin(B_\circ + \mu t + \lambda_{,}) = &(b_\circ \sin B_{,}) \cos(l_\circ - \Omega + \lambda_{,} + \mu_{,}t) \\
&+ (b_\circ \cos B_{,}) \sin(l_\circ - \Omega + \lambda_{,} + \mu\, t).
\end{aligned}$$

But the coefficient of $-\delta$I is $\sin(l_\circ - \Omega + \lambda_1)$, so that the two angles differ only by $\mu_{,}t$. As the period of this difference is about twenty-four years, it is obvious that observations extending only over a year or two do not allow the satisfactory determination of both unknown quantities δI and $b_\circ \cos B_{,}$, and that one of the two must remain practically indeterminate.

A troublesome difficulty arises from the necessity of repeating a portion of the computations, in case the adopted value f is not sufficiently near the true one. Wichmann's first value $f = 0{\cdot}2603$ was derived from a preliminary solution of his longitude-equations, and it served, together with Nicollet's value of

$I = 1° 28' 47''$, in the further computations. Wichmann's final deductions gave

$$I = 1° 32' 9''; \quad f = \frac{789''.7}{1884''} = 0.419,$$

or, neglecting $a_\circ$ and $b_\circ$ on account of the uncertainty of their determination,

$$I = 1° 32' 24''; \quad f = \frac{838''.6}{1884''} = 0.445.$$

It is to be regretted that Hartwig, instead of adopting one of these latter values, has retained in his computations the preliminary inaccurate approximations. His own final result (*Monthly Notices* for May, p. 376),

$$I = 1° 36' 39''; \quad f = 0.507,$$

gives for $\nu_\circ$ the value $1458''.5$, or the period 888·6 days. Wichmann's (last) result gives $1520''.9$, or period 852·1 days, while the preliminary value employed is $1728''.9$, or the period 749·6 days. How far the determination of $a_\circ$ is affected by the error of the assumed value of $\nu_\circ$ is at present unknown; but it need scarcely be more than hinted that the error affects most seriously the connection between the results of the observations of 1845 and of 1878. Bessel points out in Article 6 of his paper that, in case a sensible value of $a_\circ$ should be successfully ascertained, the determination of the duration of its period would afford the surest means for the exact determination of f. According to the data on page 376 the value of $a_\circ$ is found from

Wichmann's Observations	=	$183''.2 \pm 80''.6$
Hartwig's Observations	=	62.3 ± 41.9.

It is to be hoped that before long the publication of the investigation of the Moon's libration which has been undertaken at the Oxford University Observatory will facilitate some decision of what are the best available values of I and f.

In a second paper I shall treat the problem of determining the physical libration by observations of the changes which it produces in the relative position of spots; I shall, further, show the effect of the libration upon the observations of a spot in right ascension and declination, and also consider some other questions.